Hamilton-Jacobi-Bellman equation on the Wasserstein Space $\mathscr{P}_2(\mathbb{R}^d)$

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This talk is devoted to the Hamilton-Jacobi-Bellman (HJB) equation associated to the Mayer type optimal control problem on the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$ of Borel probability measures:

$$\text{minimize } g(\mu(1))$$

over all the solutions defined on the time interval $[0,1]$ of the continuity equation

$$\partial_t \mu(t) + \text{div}\,(\mathrm{f}(\mu(\mathrm{t}), \mathrm{u}(\mathrm{t})) \cdot \mu(\mathrm{t})) = 0,\ \mu(0) = \mu_0,\ \ \mathrm{u}(\mathrm{t}) \in \mathrm{U}$$

with μ_0 having a compact support. In the above $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to Lip(\mathbb{R}^d, \mathbb{R}^d)$, U is a compact metric space and $Lip(\mathbb{R}^d, \mathbb{R}^d)$ denotes the space of bounded Lipschitz functions from $\mathbb{R}^d$ into itself.

Solutions to (HJB) are defined in terms of the Hadamard type sub/superdifferentials and, under Lipschitz-like assumptions, the value function of the Mayer problem is such solution of (HJB). Continuous solutions are unique whenever we focus our attention on solutions defined on explicitly described time dependent compact valued tubes of probability measures.

We also discuss some viability and invariance theorems in the Wasserstein space and introduce a new notion of proximal normal.

References

[1] Badreddine Z. & Frankowska H. (2022) *Solutions to Hamilton-Jacobi equation on a Wasserstein space*, Calculus of Variations and PDEs

[2] Badreddine Z. & Frankowska H. (2022) *Viability and invariance of systems on metric spaces*, Nonlinear Analysis

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